

# Sources, Potentials and Fields in Lorenz and Coulomb Gauge: Cancellation of Instantaneous Interactions for Moving Point Charges

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We investigate the coupling of the electromagnetic sources (charge and current densities) to the scalar and vector potentials in classical electrodynamics, using Green function techniques. As is well known, the scalar potential shows an action-at-a-distance behavior in Coulomb gauge. The conundrum generated by the instantaneous interaction has intrigued physicists for a long time. Starting from the differential equations that couple the sources to the potentials, we here show in a concise derivation, using the retarded Green function, how the instantaneous interaction cancels in the calculation of the electric field. The time derivative of a specific additional term in the vector potential, present only in Coulomb gauge, yields a supplementary contribution to the electric field which cancels the gradient of the instantaneous Coulomb gauge scalar potential, as required by gauge invariance. This completely eliminates the contribution of the instantaneous interaction from the electric field. It turns out that a careful formulation of the retarded Green function, inspired by field theory, is required in order to correctly treat boundary terms in partial integrations. Finally, compact integral representations are derived for the Liénard–Wiechert potentials (scalar and vector) in Coulomb gauge which manifestly contain two compensating action-at-a-distance terms.

## I. INTRODUCTION

The instantaneous coupling [1] of the electrostatic potential to the charge density in Coulomb gauge (“action at a distance”) has given rise to a number of concerns, recorded in the literature. Rohrlich has addressed the issue in Ref. [2]. He argues that the seemingly instantaneous integral for the scalar potential can be rewritten as an integral involving the retarded Green function, with a nonstandard source term that does not only involve the charge density but also the longitudinal part of the current density. Using the special form of the equations that relate the charge density to the longitudinal part of the current density in Coulomb gauge, the retarded integral may be shown to be equal [2] to the instantaneous integral, thus clarifying that the instantaneous nature of the action-at-a-distance formula is a particular feature of the Coulomb gauge that does not contradict the causality principle. The problem can be sidestepped if one formulates electromagnetic theory purely in terms of fields rather than potentials, using Dirac-like equations [3], but one might otherwise argue that the gauge potentials which enter the nontrivial phase of the wave function in the Aharonov–Bohm effect, are assigned some kind of physical meaning within the concept of covariant derivatives in field theory, even if one can otherwise express the Aharonov–Bohm phase in terms of the magnetic flux entering the loop [4, 5].

One could argue that despite Rohrlich’s arguments [2], the instantaneous form of the scalar potential still poses potential problems with regard to causality, even if it is possible to rewrite it in retarded form. Namely, it could be argued that it is still not clear how the instantaneous part of the potential actually cancels when the electric and magnetic fields are calculated, even though it is possible to give an alternative, retarded form for the integral defining the scalar potential. Indeed, it would be somewhat discomfiting if the action-at-a-distance part of the interaction were to prevail in the final result for the field strength. In Ref. [6], Jackson investigates this problem in the framework of gauge potentials and field equations.

We here attempt to provide an alternative view of the problem, inspired by field theory, where the retarded Green function is closely related to the commutator of the fields [see Eq. (2.55) of [7]]. We start from the structure of the equations that couple the sources and potentials in Coulomb gauge, solve these using the retarded Green function, and show the expected cancellation of instantaneous terms in the observable electric and magnetic fields. The correct and, in our case, required formulation of the retarded Green function in space-time coordinates reads as

$$G_R(\vec{r}, t, \vec{r}', t') = \frac{c}{4\pi\epsilon_0} \frac{\Theta(t-t')}{|\vec{r}-\vec{r}'|} \left[ \delta(|\vec{r}-\vec{r}'| - c(t-t')) - \delta(|\vec{r}-\vec{r}'| + c(t-t')) \right], \quad (1)$$

where  $\Theta$  is the step function,  $c$  is the speed of light, the coordinates are  $\vec{r}$  and  $\vec{r}'$ , and the times are  $t$  and  $t'$ . We keep *both* terms in square brackets until the final stage of the calculation. In particular, we notice that the expression (1) formally is at variance with Eq. (31.45) of Schwinger’s well-known textbook [8], where the term proportional to  $\Theta(t-t')\delta(|\vec{r}-\vec{r}'| + c(t-t'))$  is discarded. In many case (some physicists would say in *most* cases), it is indeed possible to discard said term [see Eq. (23) below], as done in Ref. [8]. However, in any representation of the Dirac  $\delta$  distribution, e.g., formulated in terms of normalized Gaussians whose width tend to zero, the step function will assume values of unity within the width of the representation of the Dirac  $\delta$  function when  $|\vec{r}-\vec{r}'| \rightarrow 0^+$  and  $c(t-t') \rightarrow 0^+$ .

One thus cannot discard any term in Eq. (1) if one would like to carry out partial integrations correctly; the latter are useful in the investigation of the coupling of the sources to the potentials and fields.

The general results derived in the current work are then verified on the basis of a concrete problem, namely, the potentials generated by a moving point charge. Due to the singular character of the source terms in this problem, a treatment using the retarded Green is seen to be more convenient. In classical electrodynamics, all electric and magnetic fields can in principle be described as a superposition of fields generated by infinitesimal moving point charges. Again, the intriguing problem is that in Coulomb gauge (radiation gauge), the scalar potential  $\Phi$  can be written as an instantaneous integral over charges that are far away from the observation point. A change in a charge distribution displaced by even astronomical distance scales therefore leads to an instantaneous change in the scalar potential at an observation point on Earth. In order to address this latter question, we here not only calculate the scalar potential, but also the vector potential generated by a moving charge, in Coulomb gauge and in general form. This problem has independently attracted some interest [9–11].

In this article, we use SI mksA units throughout. We start with a number of general considerations in Sec. II, before calculating the potentials in Coulomb gauge (Sec. III). Particular emphasis is laid on the correct use of the full space-time representation of the retarded Green function, which is crucial for the cancellation of a few singular terms. Conclusions are reserved for Sec. IV.

## II. GENERAL CONSIDERATIONS

### A. Potentials and Sources

In electrodynamics, we differentiate the Lorentz force and the Lorentz transformation from the Lorenz gauge. The Lorenz gauge is Lorentz-covariant [12]. The Lorenz condition for the scalar potential  $\Phi$  and the vector potential  $\vec{A}$  reads

$$\vec{\nabla} \cdot \vec{A}_L(\vec{r}, t) + \frac{1}{c^2} \frac{\partial}{\partial t} \Phi_L(\vec{r}, t) = 0. \quad (2a)$$

The subscript  $L$  will be used throughout this article in order to denote the Lorenz gauge. The potentials are coupled to the sources by inhomogeneous wave functions,

$$\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \right) \Phi_L(\vec{r}, t) = \frac{1}{\epsilon_0} \rho(\vec{r}, t), \quad (2b)$$

$$\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \right) \vec{A}_L(\vec{r}, t) = \mu_0 \vec{J}(\vec{r}, t), \quad (2c)$$

where  $\epsilon_0$  and  $\mu_0$  are the vacuum permittivity and vacuum permeability, respectively. The charge density is  $\rho(\vec{r}, t)$ , and the current density is  $\vec{J}(\vec{r}, t)$ . In Lorenz gauge, all potentials and fields manifestly propagate with the speed of light in vacuum  $c$ . It is possible to define gauges, where the potentials propagate with different speeds [6]. For quantum electrodynamics, this is investigated in Ref. [13].

In the Coulomb gauge (subscript  $C$ ), or radiation gauge, the gauge condition is

$$\vec{\nabla} \cdot \vec{A}_C(\vec{r}, t) = 0, \quad \vec{A}_C(\vec{r}, t) = \vec{A}_{C\perp}(\vec{r}, t), \quad (3a)$$

which states that the Coulomb gauge vector potential  $\vec{A}$  is equal to its transverse component  $\vec{A}_{\perp}$ . The coupling to the sources is governed by the following equations,

$$\vec{\nabla}^2 \Phi_C(\vec{r}, t) = -\frac{1}{\epsilon_0} \rho(\vec{r}, t), \quad (3b)$$

$$\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \right) \vec{A}_{C\perp}(\vec{r}, t) = \mu_0 \vec{J}_{\perp}(\vec{r}, t), \quad (3c)$$

$$\epsilon_0 \frac{\partial}{\partial t} \vec{\nabla} \Phi_C(\vec{r}, t) = \vec{J}_{\parallel}(\vec{r}, t). \quad (3d)$$

Here, we refer to the longitudinal and transverse components of a general vector field  $\vec{J}(\vec{r}, t)$  as  $\vec{J}_{\parallel}(\vec{r}, t)$  and  $\vec{J}_{\perp}(\vec{r}, t)$ , respectively. According to a well-known theorem, any vector field can be uniquely decomposed into a longitudinal and a transverse component, as follows,

$$\vec{J}(\vec{r}, t) = \vec{J}_{\perp}(\vec{r}, t) + \vec{J}_{\parallel}(\vec{r}, t), \quad \vec{\nabla} \cdot \vec{J}_{\perp}(\vec{r}, t) = 0, \quad \vec{\nabla} \times \vec{J}_{\parallel}(\vec{r}, t) = 0. \quad (4)$$

Given  $\vec{J}(\vec{r}, t)$ , the parallel and longitudinal components can be computed as

$$\vec{J}_{\parallel}(\vec{r}, t) = -\frac{1}{4\pi} \vec{\nabla} \int \frac{\vec{\nabla}' \cdot \vec{J}(\vec{r}', t)}{|\vec{r} - \vec{r}'|} d^3 r', \quad (5a)$$

$$\vec{J}_{\perp}(\vec{r}, t) = \frac{1}{4\pi} \vec{\nabla} \times \int \frac{\vec{\nabla}' \times \vec{J}(\vec{r}', t)}{|\vec{r} - \vec{r}'|} d^3 r'. \quad (5b)$$

These are highly non-local functions of the full current density  $\vec{J}(\vec{r}, t)$ , as pointed out in Ref. [2].

## B. Potentials and Fields

The general formula for the computation of the electric field from the scalar and vector potentials is

$$\vec{E}(\vec{r}, t) = -\vec{\nabla} \Phi(\vec{r}, t) - \frac{\partial}{\partial t} \vec{A}(\vec{r}, t). \quad (6)$$

In Coulomb gauge, since  $\vec{A}_C(\vec{r}, t) = \vec{A}_{C\perp}(\vec{r}, t)$ , we can slightly rewrite this formula as

$$\vec{E}(\vec{r}, t) = -\vec{\nabla} \Phi_C(\vec{r}, t) - \frac{\partial}{\partial t} \vec{A}_{C\perp}(\vec{r}, t), \quad (7)$$

and therefore

$$\vec{E}_{\parallel}(\vec{r}, t) = -\vec{\nabla} \Phi_C(\vec{r}, t), \quad \vec{E}_{\perp}(\vec{r}, t) = -\frac{\partial}{\partial t} \vec{A}_{C\perp}(\vec{r}, t). \quad (8)$$

These components fulfill, explicitly,

$$\vec{\nabla} \times \vec{E}_{\parallel}(\vec{r}, t) = -(\vec{\nabla} \times \vec{\nabla}) \Phi_C(\vec{r}, t) = 0, \quad (9)$$

and

$$\vec{\nabla} \cdot \vec{E}_{\perp}(\vec{r}, t) = -\frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{A}_{C\perp}(\vec{r}, t) = 0. \quad (10)$$

Therefore, the full longitudinal component of the electric field is given by  $\vec{E}_{\parallel}(\vec{r}, t) = -\vec{\nabla} \Phi_C(\vec{r}, t)$ , and the transverse component in Coulomb gauge is purely given by the time derivative of the transverse vector potential, without any “admixture” from the scalar potential. In Coulomb gauge, the scalar potential  $\Phi(\vec{r}, t)$  is given by an action-at-a-distance integral, as a result of the instantaneous coupling of the potential to the source (charge density) according to Eq. (3b),

$$\Phi_C(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \int d^3 r' \frac{1}{|\vec{r} - \vec{r}'|} \rho(\vec{r}', t). \quad (11)$$

The longitudinal component of the electric field is thus calculated as

$$\vec{E}_{\parallel}(\vec{r}, t) = -\vec{\nabla} \Phi_C(\vec{r}, t) = -\frac{1}{4\pi\epsilon_0} \vec{\nabla} \int d^3 r' \frac{1}{|\vec{r} - \vec{r}'|} \rho(\vec{r}', t) \quad (12)$$

which again is an action-at-a-distance integral. The central result of the work of Rohrlich [2] (the remarks of Jefimenko [14], Heras [15] and of Rohrlich [16] do not affect the derivation) states that we can alternatively write the action-at-a-distance component of the electric field as a retarded integral,

$$\vec{E}_{\parallel}(\vec{r}, t) = - \int d^3 r' \int dt' G_R(\vec{r}, t, \vec{r}', t') \left( \frac{1}{c^2} \frac{\partial}{\partial t'} \vec{J}_{\parallel}(\vec{r}', t') + \vec{\nabla}' \rho(\vec{r}', t') \right), \quad (13)$$

where the retarded Green function  $G_R(\vec{r}, t, \vec{r}', t')$  in coordinate-space is given in Eq. (1). The retarded Green function fulfills the defining equation

$$\left( \frac{1}{c^2} \frac{\partial^2}{\partial t^2} - \vec{\nabla}^2 \right) G_R(\vec{r}, t, \vec{r}', t') = \frac{1}{\epsilon_0} \delta^{(3)}(\vec{r} - \vec{r}') \delta(t - t'), \quad (14)$$

and its explicit form has been given in Eq. (1).

Another perspective on this problem can be found by showing how the action-at-a-distance integral for the longitudinal component of the electric field actually cancels when we add the time derivative of the Coulomb-gauge vector potential. In order to find the solutions of Eqs. (2b) and (2c), we use the retarded Green function given in Eq. (1). The solutions in Lorenz gauge can be written down immediately,

$$\Phi_L(\vec{r}, t) = \int d^3r' dt' G_R(\vec{r}, t, \vec{r}', t') \rho(\vec{r}', t'), \quad (15a)$$

$$\vec{A}_L(\vec{r}, t) = \frac{1}{c^2} \int d^3r' dt' G_R(\vec{r}, t, \vec{r}', t') \vec{J}(\vec{r}', t'). \quad (15b)$$

The retarded Green function can be used in order to solve Eq. (3c), and we find the vector potential in Coulomb gauge,

$$\begin{aligned} \vec{A}_C(\vec{r}, t) &= \frac{1}{c^2} \int d^3r' \int dt' G_R(\vec{r}, t, \vec{r}', t') \vec{J}_\perp(\vec{r}', t') \\ &= \frac{1}{c^2} \int d^3r' \int dt' G_R(\vec{r}, t, \vec{r}', t') \vec{J}(\vec{r}', t') - \frac{1}{c^2} \int d^3r' \int dt' G_R(\vec{r}, t, \vec{r}', t') \vec{J}_\parallel(\vec{r}', t') \\ &= \vec{A}_L(\vec{r}, t) + \vec{A}_S(\vec{r}, t), \end{aligned} \quad (16)$$

where we have used the identity  $\vec{J}_\perp(\vec{r}', t') = \vec{J}(\vec{r}', t') - \vec{J}_\parallel(\vec{r}', t')$ , and  $\vec{A}_L(\vec{r}, t)$  (Lorenz-gauge expression) and  $\vec{A}_S(\vec{r}, t)$  are defined in the obvious way. The additional term  $\vec{A}_S(\vec{r}, t)$  is relevant to the Coulomb gauge and reads

$$\vec{A}_S(\vec{r}, t) = -\frac{1}{c^2} \int d^3r' \int dt' G_R(\vec{r}, t, \vec{r}', t') \vec{J}_\parallel(\vec{r}', t'). \quad (17)$$

The time derivative of  $\vec{A}_S(\vec{r}, t)$  contributes to the electric field,

$$\begin{aligned} E_S(\vec{r}, t) &= -\frac{\partial}{\partial t} \vec{A}_S(\vec{r}, t) = \frac{1}{c^2} \int d^3r' \int dt' \left[ \frac{\partial}{\partial t} G_R(\vec{r}, t, \vec{r}', t') \right] \vec{J}_\parallel(\vec{r}', t') \\ &= -\frac{1}{c^2} \int d^3r' \int dt' \left[ \frac{\partial}{\partial t'} G_R(\vec{r}, t, \vec{r}', t') \right] \vec{J}_\parallel(\vec{r}', t') \\ &= \frac{1}{c^2} \int d^3r' \int dt' G_R(\vec{r}, t, \vec{r}', t') \left[ \frac{\partial}{\partial t'} \vec{J}_\parallel(\vec{r}', t') \right], \end{aligned} \quad (18)$$

where we have first transformed  $\partial/\partial t \rightarrow -\partial/\partial t'$  and then used integration by parts to move the derivative on the current. Using the identity in Eq. (3d), this can be rewritten in terms of the potential as

$$\begin{aligned} E_S(\vec{r}, t) &= \epsilon_0 \int d^3r' \int dt' G_R(\vec{r}, t, \vec{r}', t') \vec{\nabla}' \left[ \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} \Phi_C(\vec{r}', t') \right] \\ &= \epsilon_0 \vec{\nabla} \int d^3r' \int dt' G_R(\vec{r}, t, \vec{r}', t') \left[ \left( \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} - \vec{\nabla}'^2 \right) + \vec{\nabla}'^2 \right] \Phi_C(\vec{r}', t') \\ &= \epsilon_0 \vec{\nabla} \int d^3r' \int dt' \left[ \left( \frac{1}{c^2} \frac{\partial^2}{\partial t'^2} - \vec{\nabla}'^2 \right) G_R(\vec{r}, t, \vec{r}', t') \right] \Phi_C(\vec{r}', t') \\ &\quad + \epsilon_0 \vec{\nabla} \int d^3r' \int dt' G_R(\vec{r}, t, \vec{r}', t') \vec{\nabla}'^2 \Phi_C(\vec{r}', t'). \end{aligned} \quad (19)$$

For the first term, we use partial integration twice, and we also take advantage of the symmetry properties of the retarded Green function. With Eqs. (3b) and (14), we find

$$\begin{aligned} E_S(\vec{r}, t) &= \epsilon_0 \vec{\nabla} \int d^3r' \int dt' \frac{1}{\epsilon_0} \delta^{(3)}(\vec{r} - \vec{r}') \delta(t - t') \Phi_C(\vec{r}', t') \\ &\quad - \epsilon_0 \vec{\nabla} \int d^3r' \int dt' G_R(\vec{r}, t, \vec{r}', t') \frac{1}{\epsilon_0} \rho(\vec{r}', t') \\ &= \vec{\nabla} \Phi_C(\vec{r}, t) - \vec{\nabla} \int d^3r' \int dt' G_R(\vec{r}, t, \vec{r}', t') \rho(\vec{r}', t') \\ &= \vec{\nabla} \Phi_C(\vec{r}, t) - \vec{\nabla} \Phi_L(\vec{r}, t). \end{aligned} \quad (20)$$

This term cancels the longitudinal electric field in Coulomb gauge  $[= -\vec{\nabla}\Phi_C(\vec{r}, t)]$  and adds the contribution to the electric field in Lorenz gauge, due to the gradient of the scalar potential  $[= -\vec{\nabla}\Phi_L(\vec{r}, t)]$ . In summary, the additional term  $-\frac{\partial}{\partial t}\vec{A}_S(\vec{r}, t)$  due to the time derivative of the added vector potential in Coulomb gauge cancels the gradient of the Coulomb gauge action-at-a-distance integral for the scalar potential and adds the Lorenz gauge gradient of the scalar potential. Otherwise, this property is implied by the fact that  $\vec{A}_S$  is necessarily equal to  $\vec{\nabla}\chi$ , where  $\chi$  is the gauge potential from the transition from Lorenz to Coulomb gauge. Here, we show the cancellation of the instantaneous interaction by first coupling the potentials to the sources, separately, in Lorenz and Coulomb gauge, using the retarded Green function, and then calculating the fields from the potentials. Our derivation is concise [Eqs. (17)–(20)].

Comparing the formulas for the electric field in Coulomb and Lorenz gauge, the following identity follows immediately,

$$\begin{aligned}\vec{E}_C(\vec{r}, t) &= -\vec{\nabla}\Phi_C(\vec{r}, t) - \frac{\partial}{\partial t}\vec{A}_C(\vec{r}, t) = -\vec{\nabla}\Phi_C(\vec{r}, t) - \frac{\partial}{\partial t}(\vec{A}_L(\vec{r}, t) + \vec{A}_S(\vec{r}, t)) \\ &= -\vec{\nabla}\Phi_C(\vec{r}, t) - \frac{\partial}{\partial t}\vec{A}_L(\vec{r}, t) + (\vec{\nabla}\Phi_C(\vec{r}, t) - \vec{\nabla}\Phi_L(\vec{r}, t)) = -\vec{\nabla}\Phi_L(\vec{r}, t) - \frac{\partial}{\partial t}\vec{A}_L(\vec{r}, t) = \vec{E}_L(\vec{r}, t).\end{aligned}\quad (21)$$

We have temporarily denoted the “Coulomb gauge” electric field as  $\vec{E}_C(\vec{r}, t)$  and the “Lorenz gauge” electric field as  $\vec{E}_L(\vec{r}, t)$ , even if both are actually equal due to gauge invariance, as shown.

Let us briefly summarize the findings. In the Coulomb gauge, the vector potential is connected only to the transverse component of the current density. We can write the transverse component of the current density as the difference of the full current density, minus the longitudinal component of the current density [Eq. (16)]. The full current density gives the same contribution to the vector potential as in Lorenz gauge [Eq. (15a)]. In Coulomb gauge, there is an additional term  $\vec{A}_S$  in the vector potential which corresponds to the negative of the longitudinal component of the current density. According to Eq. (20), it can be written as a gradient vector and therefore does not affect the result for the magnetic field, which thus is the same as in Lorenz gauge. The (negative of the) time derivative of the additional term in the vector potential yields an additional contribution to the electric field, in Coulomb gauge. The additional term in the electric field can be transformed into two parts, the first of which cancels the seemingly instantaneous electric field contribution in Coulomb gauge, obtained from the Coulomb-gauge electric potential, and the second yields the same result (the retarded one) as the gradient of the electric potential  $\Phi_L$  in Lorenz gauge. In the end, the action-at-a-distance integral cancels, and the gauge invariance of the electric field is shown.

### III. MOVING CHARGES IN DIFFERENT GAUGES

#### A. Moving Charges and Lorenz Gauge

The representation (1) of the retarded Green function in coordinate space contains two Dirac- $\delta$  functions and is highly singular. Therefore, it is certainly sensible to verify the conclusions reached thus far on the basis of a concrete problem where the actual form of the retarded Green function needs to be used (Liénard–Wiechert potentials for a moving point charge), because in addition to the Green function, the charge and current distributions also become singular. Indeed, the charge and current densities are

$$\rho(\vec{r}, t) = q \delta^{(3)}(\vec{r} - \vec{R}(t)), \quad (22a)$$

$$\vec{J}(\vec{r}, t) = q \delta^{(3)}(\vec{r} - \vec{R}(t)) \left[ \frac{d}{dt}\vec{R}(t) \right], \quad (22b)$$

where  $\vec{R}(t)$  is the particle trajectory. If we keep only the first term in square brackets in Eq. (1) and write

$$G_R(\vec{r}, t, \vec{r}', t') \approx \frac{c}{4\pi\epsilon_0} \frac{\Theta(t - t')}{|\vec{r} - \vec{r}'|} \delta(|\vec{r} - \vec{r}'| - c(t - t')), \quad (23)$$

then we can express the potentials of the moving charges in Eq. (22) as follows,

$$\Phi_L(\vec{r}, t) \approx \frac{q}{4\pi\epsilon_0} \int d^3r' dt' \left[ \delta^{(3)}(\vec{r}' - \vec{R}(t')) \right] \left\{ \frac{\Theta(t - t')}{|\vec{r} - \vec{r}'|} \delta\left(t - t' - \frac{|\vec{r} - \vec{r}'|}{c}\right) \right\}, \quad (24a)$$

$$\vec{A}_L(\vec{r}, t) \approx \frac{q}{4\pi\epsilon_0 c^2} \int d^3r' dt' \left[ \delta^{(3)}(\vec{r}' - \vec{R}(t')) \dot{\vec{R}}(t') \right] \left\{ \frac{\Theta(t - t')}{|\vec{r} - \vec{r}'|} \delta\left(t - t' - \frac{|\vec{r} - \vec{r}'|}{c}\right) \right\}. \quad (24b)$$

We now carry out the integration over  $d^3r'$ , which eliminates three integrations at once,

$$\Phi_L(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \int dt' \Theta(t - t') \frac{1}{|\vec{r} - \vec{R}(t')|} \delta\left(t' - \left(t - \frac{|\vec{r} - \vec{R}(t')|}{c}\right)\right), \quad (25a)$$

$$\vec{A}_L(\vec{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \int dt' \Theta(t - t') \frac{\dot{\vec{R}}(t')}{|\vec{r} - \vec{R}(t')|} \delta\left(t' - \left(t - \frac{|\vec{r} - \vec{R}(t')|}{c}\right)\right). \quad (25b)$$

The  $\delta$  function peaks at

$$t' = t_{\text{ret}} = t - \frac{|\vec{r} - \vec{R}(t')|}{c} = t - \frac{|\vec{r} - \vec{R}(t_{\text{ret}})|}{c} < t, \quad (26)$$

or

$$t_{\text{ret}} = t - \frac{|\vec{r} - \vec{R}(t_{\text{ret}})|}{c}, \quad c(t - t_{\text{ret}})^2 = \left(\vec{r} - \vec{R}(t_{\text{ret}})\right)^2, \quad (27)$$

so that the step function is always unity at the point where the Dirac- $\delta$  peaks. That means that all points on the trajectory of the particle for which the retardation condition is fulfilled, contribute to the integrals. Let us assume that the Dirac- $\delta$  peaks only once, namely, at  $t' = t_{\text{ret}}$ . The integration over  $dt'$  in Eq. (25) still leads to a nontrivial Jacobian as the argument of the Dirac- $\delta$  is nontrivial and needs to be differentiated with respect to  $t'$ . Indeed, we have

$$\frac{d}{dt'} \left( t' - t + \frac{|\vec{r} - \vec{R}(t')|}{c} \right) = 1 - \frac{1}{c} \frac{d\vec{R}(t')}{dt'} \cdot \frac{\vec{r} - \vec{R}(t')}{|\vec{r} - \vec{R}(t')|}, \quad (28)$$

at  $t' = t_{\text{ret}}$ . So, the well-known Liénard–Wiechert potentials in Lorenz gauge read (see Ref. [1]),

$$\Phi_L(\vec{r}, t) = \frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{R}(t_{\text{ret}})|} \left( 1 - \frac{\dot{\vec{R}}(t_{\text{ret}})}{c} \cdot \frac{\vec{r} - \vec{R}(t_{\text{ret}})}{|\vec{r} - \vec{R}(t_{\text{ret}})|} \right)^{-1}, \quad (29a)$$

$$\vec{A}_L(\vec{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\dot{\vec{R}}(t_{\text{ret}})}{|\vec{r} - \vec{R}(t_{\text{ret}})|} \left( 1 - \frac{\dot{\vec{R}}(t_{\text{ret}})}{c} \cdot \frac{\vec{r} - \vec{R}(t_{\text{ret}})}{|\vec{r} - \vec{R}(t_{\text{ret}})|} \right)^{-1}. \quad (29b)$$

Here, we write the equations in SI mksA units. *The validity of this well-known derivation, which we recall for convenience, hinges upon the fact that no partial integrations involving the simplified Green function (23) were necessary.*

## B. Coulomb Gauge and Current Components

The charge and current density corresponding to the moving point charge are given by Eq. (22), and the Coulomb-gauge coupling to the potentials is given by Eq. (3). The task is to calculate, explicitly, the Liénard–Wiechert potentials due to the moving charges, and to show the gauge invariance of the fields. In particular, the question needs to be answered how the instantaneous Coulomb interaction should be treated. The longitudinal component of the current is given by

$$\vec{J}_{\parallel}(\vec{r}, t) = \frac{q}{4\pi} \vec{\nabla} \frac{\partial}{\partial t} \frac{1}{|\vec{r} - \vec{R}(t)|}. \quad (30)$$

It is advantageous to leave this result in symbolic form. We can easily verify that  $\vec{J}_{\parallel}$  carries the entire divergence of  $\vec{J}$ , and that the curl of  $\vec{J}_{\parallel}$  vanishes. The transverse component of the point-particle current density can be written down immediately,

$$\vec{J}_{\perp}(\vec{r}, t) = \vec{J}(\vec{r}, t) - \vec{J}_{\parallel}(\vec{r}, t) = q \dot{\vec{R}}(t) \delta^{(3)}(\vec{r} - \vec{R}(t)) - \frac{q}{4\pi} \vec{\nabla} \frac{\partial}{\partial t} \frac{1}{|\vec{r} - \vec{R}(t)|}.$$

Its curl is given by

$$\vec{\nabla} \times \vec{J}_{\perp}(\vec{r}, t) = \vec{\nabla} \times \left( q \dot{\vec{R}}(t) \delta^{(3)}(\vec{r} - \vec{R}(t)) \right) = -q \dot{\vec{R}}(t) \times \vec{\nabla} \delta^{(3)}(\vec{r} - \vec{R}(t)) = \vec{\nabla} \times \vec{J}(\vec{r}, t). \quad (31)$$

### C. Scalar Potential in Coulomb Gauge

The scalar potential in Coulomb gauge is a solution of (3b) and is easily calculated as the action-at-a-distance term

$$\Phi_C(\vec{r}, t) = \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r} - \vec{R}(t)|}. \quad (32)$$

Let us try to verify the relation (3d), which is easily done as follows,

$$\epsilon_0 \vec{\nabla} \left[ \frac{\partial}{\partial t} \Phi_C(\vec{r}, t) \right] = \epsilon_0 \vec{\nabla} \frac{\partial}{\partial t} \frac{1}{4\pi\epsilon_0} \frac{q}{|\vec{r} - \vec{R}(t)|} = \frac{q}{4\pi} \vec{\nabla} \frac{\partial}{\partial t} \frac{1}{|\vec{r} - \vec{R}(t)|} = \vec{J}_{\parallel}(\vec{r}, t). \quad (33)$$

At this point, the scalar potential in Coulomb gauge is calculated as an action-at-a-distance integral. It remains to investigate the vector potential in Coulomb gauge, which is responsible for the eventual cancellation of the action-at-a-distance solution (32). This is done in the next section.

### D. Vector Potential in Coulomb Gauge

We recall that in Coulomb gauge, there is an additional term  $\vec{A}_S$  in the vector potential given by Eq. (17). In order to calculate it, we need the result (30) for the longitudinal part of the current density. We write

$$\begin{aligned} \vec{A}_S(\vec{r}, t) &= -\frac{1}{c^2} \int d^3r' \int dt' G_R(\vec{r}, t, \vec{r}', t') \frac{q}{4\pi} \vec{\nabla}' \left( \frac{\partial}{\partial t'} \frac{1}{|\vec{r}' - \vec{R}(t')|} \right) \\ &= -\frac{q}{(4\pi)^2 \epsilon_0 c} \int d^3r' \int dt' \left( \frac{\partial}{\partial t'} \left[ \frac{\vec{\nabla}'}{|\vec{r}' - \vec{R}(t')|} \right] \right) \frac{\Theta(t - t')}{|\vec{r} - \vec{r}'|} \\ &\quad \times \{ \delta(|\vec{r} - \vec{r}'| - c(t - t')) - \delta(|\vec{r} - \vec{r}'| + c(t - t')) \}. \end{aligned} \quad (34)$$

It now turns out to be very important to use the complete retarded Green function, when we perform a partial integration with respect to  $t'$  and shift the  $t'$ -derivative from the current term on the Dirac- $\delta$ . When encountering a term from the  $t'$ -derivative acting on the Heavyside- $\Theta$ , which gives  $-\delta(t - t')$ , then we shall find that it causes the two Dirac- $\delta$ s from the Green function to have the same arguments and cancel. In conclusion, the only resulting term after the partial integration with respect to  $t'$  is

$$\begin{aligned} \vec{A}_S(\vec{r}, t) &= \frac{q}{(4\pi)^2 \epsilon_0} \int dt' \int d^3r' \left( \frac{\vec{\nabla}'}{|\vec{r}' - \vec{R}(t')|} \right) \frac{\Theta(t - t')}{|\vec{r} - \vec{r}'|} \\ &\quad \times \{ \delta'(|\vec{r} - \vec{r}'| - c(t - t')) + \delta'(|\vec{r} - \vec{r}'| + c(t - t')) \} \\ &= \frac{q}{(4\pi)^2 \epsilon_0} \int dt' \Theta(t - t') \int d^3\xi \left( \frac{\partial}{\partial \xi} \frac{1}{|\vec{\xi} - (\vec{r} - \vec{R}(t'))|} \right) \frac{1}{|\vec{\xi}|} \\ &\quad \times \left\{ \delta'(|\vec{\xi}| - c(t - t')) + \delta'(|\vec{\xi}| + c(t - t')) \right\}. \end{aligned} \quad (35)$$

The sign of the final result is clear from the fact that we differentiate the Dirac- $\delta$ s with respect to  $t'$ , not  $t$ . We have defined the new variable  $\vec{\xi} = \vec{r} - \vec{r}'$ , with  $d\xi = -dr'$  and  $\vec{\nabla}' = -\partial/\partial\vec{\xi}$ . Now, we can change the variable of the gradient yet again to  $\vec{r}$  and move it out of the integral. We define

$$\vec{x}(t') = \vec{r} - \vec{R}(t') \quad (36)$$

and obtain

$$\vec{A}_S(\vec{r}, t) = \frac{q}{(4\pi)^2 \epsilon_0} \vec{\nabla} \int dt' \Theta(t - t') \int d^3\xi \frac{1}{|\vec{\xi} - \vec{x}(t')|} \frac{1}{|\vec{\xi}|} \left\{ \delta'(|\vec{\xi}| - c(t - t')) + \delta'(|\vec{\xi}| + c(t - t')) \right\}. \quad (37)$$

Now, we expand the first of the terms into spherical harmonics, using the well-known formula

$$\frac{1}{|\vec{\xi} - \vec{x}(t')|} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \frac{4\pi}{2\ell+1} \frac{r_{<}^{\ell}}{r_{>}^{\ell+1}} Y_{\ell m}(\Omega_{\xi}) Y_{\ell m}^*(\Omega_x), \quad r_{<} = \min(|\vec{\xi}|, x(t')), \quad r_{>} = \max(|\vec{\xi}|, x(t')), \quad (38)$$



where  $\Omega_\xi$  and  $\Omega_x$  denote the solid angles for the vector variables  $\vec{\xi}$  and  $\vec{x}$ , and obtain

$$\begin{aligned} \vec{A}_S(\vec{r}, t) = & \frac{q}{(4\pi)^2 \epsilon_0} \vec{\nabla} \int dt' \Theta(t-t') \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \int d\Omega_\xi \frac{4\pi}{2\ell+1} Y_{\ell m}(\Omega_\xi) Y_{\ell m}^*(\Omega_x) \left[ \int_0^{x(t')} d\xi \xi^2 \frac{1}{\xi} \frac{\xi^\ell}{x(t')^{\ell+1}} \right. \\ & \times \left. \{ \delta'(\xi - c(t-t')) + \delta'(\xi + c(t-t')) \} + \int_{x(t')}^{\infty} d\xi \xi^2 \frac{1}{\xi} \frac{x(t')^\ell}{\xi^{\ell+1}} \{ \delta'(\xi - c(t-t')) + \delta'(\xi + c(t-t')) \} \right], \end{aligned} \quad (39)$$

where we use  $|\vec{\xi}| = \xi$  as well as  $|\vec{x}(t')| = x(t')$ , and split the integral according to the  $r_<$  versus  $r_>$  symbols in Eq. (38). The symbol  $x(t') = |\vec{x}(t')| = |\vec{r} - \vec{R}(t')|$  is not being integrated over and therefore pulled out from the integrand into the prefactor. Because  $1/|\vec{\xi}|$  is a scalar, only the  $\ell = 0$ ,  $m = 0$  component of the multipole expansion contributes, and the expression simplifies to

$$\begin{aligned} \vec{A}_S(\vec{r}, t) = & \frac{q}{4\pi\epsilon_0} \vec{\nabla} \int dt' \Theta(t-t') \left[ \frac{1}{x(t')} \int_0^{x(t')} d\xi \xi \left\{ \frac{d}{d\xi} \delta(\xi - c(t-t')) + \frac{d}{d\xi} \delta(\xi + c(t-t')) \right\} \right. \\ & \left. + \int_{x(t')}^{\infty} d\xi \left\{ \frac{d}{d\xi} \delta(\xi - c(t-t')) + \frac{d}{d\xi} \delta(\xi + c(t-t')) \right\} \right]. \end{aligned} \quad (40)$$

Here, we have written  $\delta'$  as a derivative with respect to  $\xi$ . The first term in square brackets is now integrated by parts, whereas the last term can be integrated directly because it constitutes a total differential. The result is

$$\begin{aligned} \vec{A}_S(\vec{r}, t) = & \frac{q}{4\pi\epsilon_0} \vec{\nabla} \int dt' \Theta(t-t') \left\{ \left[ \frac{\xi}{x(t')} \{ \delta(\xi - c(t-t')) + \delta(\xi + c(t-t')) \} \right]_0^{x(t')} \right. \\ & \left. - \frac{1}{x(t')} \int_0^{x(t')} d\xi \{ \delta(\xi - c(t-t')) + \delta(\xi + c(t-t')) \} + \left[ \delta(\xi - c(t-t')) + \delta(\xi + c(t-t')) \right]_{x(t')}^{\infty} \right\}. \end{aligned} \quad (41)$$

After the cancellation of boundary terms (the limits must be considered carefully), the expression reads

$$\begin{aligned} \vec{A}_S(\vec{r}, t) = & -\frac{q}{4\pi\epsilon_0} \vec{\nabla} \int dt' \Theta(t-t') \frac{1}{x(t')} \int_0^{x(t')} d\xi \{ \delta(\xi - c(t-t')) + \delta(\xi + c(t-t')) \} \\ = & -\frac{q}{4\pi\epsilon_0} \vec{\nabla} \int_{-\infty}^t dt' \frac{1}{|\vec{r} - \vec{R}(t')|} \int_0^{|\vec{r} - \vec{R}(t')|} d\xi \{ \delta(\xi - c(t-t')) + \delta(\xi + c(t-t')) \}. \end{aligned} \quad (42)$$

We now use the same approximation (23) for the retarded Green function as in Lorenz gauge. The second of the Dirac- $\delta$ s only peaks for  $t = t'$  within the  $t'$  integration, and we can discard this contribution because the integration limits for the  $t'$  integration ensure that  $t'$  cannot become larger than  $t$ . On the other hand, if  $t'$  is smaller than  $t_{\text{ret}} = t - |\vec{r} - \vec{R}(t')|/c$ , then the first Dirac- $\delta$  also cannot peak because the  $\xi$  integration stops at the upper limit of  $\xi = |\vec{r} - \vec{R}(t')|$ . The net effect of the  $\xi$  integration thus is that the  $t'$  integration will be restricted to the interval  $t' \in (t_{\text{ret}}, t)$ ,

$$\vec{A}_S(\vec{r}, t) = -\frac{q}{4\pi\epsilon_0} \vec{\nabla} \int_{t_{\text{ret}}}^t dt' \frac{1}{|\vec{r} - \vec{R}(t')|}. \quad (43)$$

We have thus obtained, in symbolic form, the vector potential for a moving particle in the Coulomb gauge. We remember that according to Eq. (16), the full Coulomb-gauge vector potential is given by  $\vec{A}_C(\vec{r}, t) = \vec{A}_L(\vec{r}, t) + \vec{A}_S(\vec{r}, t)$ ,



where  $\vec{A}_L$  is given in Eq. (15b) and for a moving point particle in Eq. (29b). So, the Liénard-Wiechert potentials in Coulomb gauge are

$$\Phi_C(\vec{r}, t) = \underbrace{\frac{q}{4\pi\epsilon_0} \frac{1}{|\vec{r} - \vec{R}(t)|}}_{\text{action-at-a-distance}}, \quad (44a)$$

$$\vec{A}_C(\vec{r}, t) = \frac{q}{4\pi\epsilon_0 c^2} \frac{\dot{\vec{R}}(t_{\text{ret}})}{|\vec{r} - \vec{R}(t_{\text{ret}})|} \left( 1 - \frac{\dot{\vec{R}}(t_{\text{ret}})}{c} \cdot \frac{\vec{r} - \vec{R}(t_{\text{ret}})}{|\vec{r} - \vec{R}(t_{\text{ret}})|} \right)^{-1} - \underbrace{\frac{q}{4\pi\epsilon_0} \vec{\nabla} \int_{t_{\text{ret}}}^t dt' \frac{1}{|\vec{r} - \vec{R}(t')|}}_{\text{action-at-a-distance}}. \quad (44b)$$

To the best of our knowledge, these compact integral representations have not yet appeared in the literature for the Liénard-Wiechert potentials in Coulomb gauge. These equations illustrate the instantaneous character of the scalar potential but also the existence of an additional instantaneous term in the vector potential, whose calculation necessitates the knowledge of the trajectory of the particle over the time interval  $t' \in (t_{\text{ret}}, t)$ . The latter term is responsible for the required cancellation of the non-causal terms in the calculation of the observable electric and magnetic fields.

The general cancellation mechanism given by Eq. (20) can be verified as follows. We recall that  $t_{\text{ret}}$  is an implicit function of the form

$$F(t, t_{\text{ret}}) = t - t_{\text{ret}} - \frac{|\vec{r} - \vec{R}(t_{\text{ret}})|}{c} = 0. \quad (45)$$

The implicit function theorem then states that the derivative of  $t_{\text{ret}}$  with respect to  $t$  is

$$\frac{\partial t_{\text{ret}}}{\partial t} = - \left( \frac{\partial F(t, t_{\text{ret}})}{\partial t_{\text{ret}}} \right)^{-1} \frac{\partial F(t, t_{\text{ret}})}{\partial t} = \left( 1 - \frac{\dot{\vec{R}}(t_{\text{ret}})}{c} \cdot \frac{\vec{r} - \vec{R}(t_{\text{ret}})}{|\vec{r} - \vec{R}(t_{\text{ret}})|} \right)^{-1}, \quad (46)$$

a result which can alternatively be derived according to Eq. (A3) below. The derivative of  $\vec{A}_S$  with respect to  $t$  is thus found as

$$\begin{aligned} -\frac{\partial}{\partial t} \vec{A}_S(\vec{r}, t) &= \frac{q}{4\pi\epsilon_0} \vec{\nabla} \frac{\partial}{\partial t} \int_{t_{\text{ret}}}^t dt' \frac{1}{|\vec{r} - \vec{R}(t')|} \\ &= \frac{q}{4\pi\epsilon_0} \vec{\nabla} \frac{1}{|\vec{r} - \vec{R}(t)|} - \frac{q}{4\pi\epsilon_0} \vec{\nabla} \frac{1}{|\vec{r} - \vec{R}(t_{\text{ret}})|} \frac{\partial t_{\text{ret}}}{\partial t} \\ &= \frac{q}{4\pi\epsilon_0} \vec{\nabla} \frac{1}{|\vec{r} - \vec{R}(t)|} - \frac{q}{4\pi\epsilon_0} \vec{\nabla} \left[ \frac{1}{|\vec{r} - \vec{R}(t_{\text{ret}})|} \left( 1 - \frac{\dot{\vec{R}}(t_{\text{ret}})}{c} \cdot \frac{\vec{r} - \vec{R}(t_{\text{ret}})}{|\vec{r} - \vec{R}(t_{\text{ret}})|} \right)^{-1} \right] \\ &= \vec{\nabla} \Phi_C(\vec{r}, t) - \vec{\nabla} \Phi_L(\vec{r}, t). \end{aligned} \quad (47)$$

In Appendix A, we verify that the result given in Eq. (44) is consistent with the general formula given in Eq. (3.10) of Ref. [6] for potentials in Coulomb gauge. While our result could be derived from Eq. (3.10) of Ref. [6], this alternative derivation is not immediate; we present a detailed derivation in Eq. (A8) below which illustrates the mechanism by which the action of the  $\vec{\nabla}$  operator on the action-at-a-distance integral in Eq. (44b) generates nontrivial boundary terms from the differentiation of the lower integration limit  $t' = t_{\text{ret}}$ . Our treatment using the full retarded Green function simplifies the inclusion of the nontrivial Jacobian from the time integration.

#### IV. CONCLUSIONS

The main theme of the current investigation is to show that the action-at-a-distance solution (44a) of the Coulomb gauge scalar potential coupled to the charge density, does not cause any problems with respect to the principle of causality. In order to provide for a concrete framework for the calculations, we consider the scalar and vector potentials corresponding to a moving point particle given in Eqs. (22a) and (22b).

According to Eq. (3c), the vector potential in Coulomb gauge is transverse and it couples only to the transverse component of the current density. Furthermore, according to Eq. (16), the Coulomb gauge vector potential can be written as the difference of the full Lorenz gauge vector potential, minus the vector potential generated by the longitudinal part of the current density. The time derivative of the latter component of the vector potential, generated by the longitudinal component of the current density, yields an additional term for the electric field, which supplements the gradient of the (instantaneous) action-at-a-distance integral from the charge density. We show for the general case as well as by an explicit calculation for the moving point charge that the time derivative of the additional term in the vector potential in Coulomb gauge can be written as the sum of two terms. The first term cancels the action-at-a-distance integral, while the second is equal to the gradient of the Lorenz gauge scalar potential. This is summarized in Eq. (21).

Thus, the electric field in Coulomb gauge is shown to be equal to the electric field in Lorenz gauge, as it should be. Moreover, our calculation provides for an additional perspective on the problem: namely, the instantaneous integral over the charge density, whose gradient contributes to the electric field in Coulomb gauge, actually cancels against the additional time derivative from the supplement to the vector potential in Coulomb gauge.

In addition to Rohrlich's arguments [2], we are able to say that the instantaneous action-at-a-distance integral for the scalar potential cancels against a term generated by the time derivative of the vector potential in Coulomb gauge and thus *does not contribute* to the physically observable electric field. The conclusions of Ref. [2] and of the current work can thus be summarized as follows. The action-at-a-distance integral (11) for the Coulomb gauge scalar potential can (i) be rewritten in terms of a retarded integral, over different source terms, which happens to be equal to the instantaneous interaction integral by virtue of the particular properties of the Coulomb gauge, and (ii) does not contribute to the electric field because its gradient is precisely canceled by an additional contribution to the time derivative of the vector potential present only in Coulomb gauge.

Compact integral expressions for the Coulomb-gauge potentials generated by a moving charge (the ‘‘Coulomb-gauge Liénard–Wiechert potentials’’) are written down in Eq. (44). They are consistent with the general treatment of Ref. [6] and generalize earlier work on this problem [9–11], where results were obtained for specific kinematic conditions (e.g., uniform motion).

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### Appendix A: Verification of the Liénard–Wiechert Potential in Coulomb Gauge

We compare to the treatment outlined in Ref. [6]. First, we need a few nontrivial derivatives. The retarded time fulfills the equation

$$t_{\text{ret}} = t - \frac{|\vec{r} - \vec{R}(t_{\text{ret}})|}{c}. \quad (\text{A1})$$

Under the variation  $t \rightarrow \delta t$  and  $t_{\text{ret}} \rightarrow t_{\text{ret}} + \delta t_{\text{ret}}$ , keeping Eq. (A1) fulfilled, we have

$$t_{\text{ret}} + \delta t_{\text{ret}} = t + \delta t - \frac{|\vec{r} - \vec{R}(t_{\text{ret}} + \delta t_{\text{ret}})|}{c} = t + \delta t + \delta t_{\text{ret}} \dot{\vec{R}}(t_{\text{ret}}) \cdot \vec{\nabla} \frac{|\vec{r} - \vec{R}(t_{\text{ret}})|}{c}. \quad (\text{A2})$$

Solving for  $t_{\text{ret}}$ , we find

$$\frac{\partial t_{\text{ret}}}{\partial t} = \frac{\delta t_{\text{ret}}}{\delta t} = \left( 1 - \dot{\vec{R}}(t_{\text{ret}}) \cdot \vec{\nabla} \frac{|\vec{r} - \vec{R}(t_{\text{ret}})|}{c} \right)^{-1}. \quad (\text{A3})$$

By contrast, varying  $\vec{r} \rightarrow \vec{r} + \delta \vec{r}$  and  $t_{\text{ret}} \rightarrow t_{\text{ret}} + \delta t_{\text{ret}}$ , we obtain

$$\begin{aligned} t_{\text{ret}} + \delta t_{\text{ret}} &= t - \frac{|\vec{r} + \delta \vec{r} - \vec{R}(t_{\text{ret}} + \delta t_{\text{ret}})|}{c} \\ &= t - \delta \vec{r} \cdot \vec{\nabla} \frac{|\vec{r} - \vec{R}(t_{\text{ret}})|}{c} + \delta t_{\text{ret}} \dot{\vec{R}}(t_{\text{ret}}) \cdot \vec{\nabla} \frac{|\vec{r} - \vec{R}(t_{\text{ret}})|}{c}, \end{aligned} \quad (\text{A4})$$

and thus

$$\vec{\nabla} t_{\text{ret}} = \frac{\delta t_{\text{ret}}}{\delta \vec{r}} = -\frac{\vec{r} - \vec{R}(t_{\text{ret}})}{c |\vec{r} - \vec{R}(t_{\text{ret}})|} \left( 1 - \frac{\dot{\vec{R}}(t_{\text{ret}}) \cdot \vec{\nabla} |\vec{r} - \vec{R}(t_{\text{ret}})|}{c} \right)^{-1}. \quad (\text{A5})$$

The retarded character of the interaction is still visible. We now refer to Ref. [6]. The variable  $R = |\vec{x} - \vec{x}'|$  is defined in the text after Eq. (2.5) of Ref. [6] and we use this identification of  $\vec{x}$  in the following instead of the definition given in Eq. (36). The time  $t'$  in Jackson's formulas is always [also see text after Eq. (2.5) of Ref. [6]]

$$t_{\text{ret}} = t' = t - R/c = t - |\vec{x} - \vec{x}'|/c = t - |\vec{x} - \vec{R}(t_{\text{ret}})|/c, \quad (\text{A6})$$

and thus defined by an implicit equation. Then, according to Eq. (3.10) of Ref. [6],

$$\begin{aligned} 4\pi\epsilon_0 c \vec{A}_C(\vec{r}, t) &= \frac{1}{c} \int d^3 x' \left( \frac{1}{R} \left[ \vec{J}(\vec{x}', t') - c \hat{R} \rho(\vec{x}', t') \right]_{t'=t_{\text{ret}}} \right) + c \int d^3 x' \frac{\hat{R}}{R^2} \int_0^{R/c} d\tau \rho(\vec{x}', t - \tau) \\ &= \frac{1}{c} \int d^3 r' \frac{1}{|\vec{r} - \vec{r}'|} \vec{J}(\vec{r}', t_{\text{ret}}) - \int d^3 r' \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^2} \rho(\vec{r}', t_{\text{ret}}) + c \int_{t_{\text{ret}}}^t d\tau \int d^3 r' \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \rho(\vec{r}', \tau), \end{aligned} \quad (\text{A7})$$

where  $4\pi\epsilon_0 c$  is restored in the SI mksA system we use here. The integral (A7) can immediately be applied to non-singular charge and current distributions. However, for a point charge, we have to plug the singular expressions for the charge and current densities (22) into Eq. (A7), so that we cannot carry out the  $d^3 r'$  integrations immediately because  $t_{\text{ret}}$  itself is a function of  $\vec{r}'$ . It is convenient to re-introduce the  $t'$  integrations, then carry out the  $d^3 r'$  integrations, and then, carry out the  $t'$  integrations which lead to the Jacobians. The derivation is rather straightforward but a detailed discussion necessitates a number of intermediate steps, which we would like to present in detail,

$$\begin{aligned} 4\pi\epsilon_0 c \vec{A}_C(\vec{r}, t) &= \frac{1}{c} \int d^3 r' \frac{1}{|\vec{r} - \vec{r}'|} q \left. \frac{\partial \vec{R}(t')}{\partial t'} \right|_{t=t_{\text{ret}}} \delta^{(3)}(\vec{r}' - \vec{R}(t_{\text{ret}})) - \int d^3 r' \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^2} q \delta^{(3)}(\vec{r}' - \vec{R}(t_{\text{ret}})) \\ &\quad + c \int_{t_{\text{ret}}}^t d\tau \int d^3 r' \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} q \delta^{(3)}(\vec{r}' - \vec{R}(t_{\text{ret}})) \\ &= \frac{1}{c} \int d^3 r' \int dt' \frac{1}{|\vec{r} - \vec{r}'|} q \frac{\partial \vec{R}(t')}{\partial t'} \delta^{(3)}(\vec{r}' - \vec{R}(t')) \delta\left(t' - t + \frac{|\vec{r} - \vec{r}'|}{c}\right) \\ &\quad - \int d^3 r' \int dt' \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^2} q \delta^{(3)}(\vec{r}' - \vec{R}(t')) \delta\left(t' - t + \frac{|\vec{r} - \vec{r}'|}{c}\right) \\ &\quad + q c \int_{t_{\text{ret}}}^t d\tau \int d^3 r' \frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|^3} \delta^{(3)}(\vec{r}' - \vec{R}(t_{\text{ret}})) \\ &= \frac{q}{c} \int dt' \frac{1}{|\vec{r} - \vec{R}(t')|} \frac{\partial \vec{R}(t')}{\partial t'} \delta\left(t' - t + \frac{|\vec{r} - \vec{R}(t')|}{c}\right) \\ &\quad - q \int dt' \frac{\vec{r} - \vec{R}(t')}{|\vec{r} - \vec{R}(t')|^2} \delta\left(t' - t + \frac{|\vec{r} - \vec{R}(t')|}{c}\right) + q c \int_{t_{\text{ret}}}^t d\tau \frac{\vec{r} - \vec{R}(\tau)}{|\vec{r} - \vec{R}(\tau)|^3} \\ &= \frac{q}{c} \frac{\dot{\vec{R}}(t_{\text{ret}})}{|\vec{r} - \vec{R}(t_{\text{ret}})|} \left( 1 - \frac{\dot{\vec{R}}(t_{\text{ret}})}{c} \cdot \frac{\vec{r} - \vec{R}(t_{\text{ret}})}{|\vec{r} - \vec{R}(t_{\text{ret}})|} \right)^{-1} \\ &\quad - q \frac{\vec{r} - \vec{R}(t_{\text{ret}})}{|\vec{r} - \vec{R}(t_{\text{ret}})|^2} \left( 1 - \frac{\dot{\vec{R}}(t_{\text{ret}})}{c} \cdot \frac{\vec{r} - \vec{R}(t_{\text{ret}})}{|\vec{r} - \vec{R}(t_{\text{ret}})|} \right)^{-1} + q c \int_{t_{\text{ret}}}^t d\tau \frac{\vec{r} - \vec{R}(\tau)}{|\vec{r} - \vec{R}(\tau)|^3} \\ &= \frac{q}{c} \frac{\dot{\vec{R}}(t_{\text{ret}})}{|\vec{r} - \vec{R}(t_{\text{ret}})|} \frac{\partial t_{\text{ret}}}{\partial t} + q c \frac{1}{|\vec{r} - \vec{R}(t_{\text{ret}})|} \vec{\nabla} t_{\text{ret}} - q c \int_{t_{\text{ret}}}^t d\tau \vec{\nabla} \frac{1}{|\vec{r} - \vec{R}(\tau)|} \\ &= \frac{q}{c} \frac{\dot{\vec{R}}(t_{\text{ret}})}{|\vec{r} - \vec{R}(t_{\text{ret}})|} \left( 1 - \frac{\dot{\vec{R}}(t_{\text{ret}})}{c} \cdot \frac{\vec{r} - \vec{R}(t_{\text{ret}})}{|\vec{r} - \vec{R}(t_{\text{ret}})|} \right)^{-1} - q c \vec{\nabla} \int_{t_{\text{ret}}}^t d\tau \frac{1}{|\vec{r} - \vec{R}(\tau)|}. \end{aligned} \quad (\text{A8})$$

The expression in the last line is finally equal to the result given in Eq. (44). The results (28), (A3) and (A5) have been used at various places during the derivation.

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